Explicit Blowing-up Solutions to the Schrödinger Maps from $\mathbf{R}^2$ to the Hyperbolic 2-Space $\mathbf{H}^2$

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Abstract

In this article, we prove that the equation of the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ is $SU(1,1)$-gauge equivalent to the following $1+2$ dimensional nonlinear Schrödinger-type system of unknown three complex functions p, q, r and a real function u:

$$\begin{cases} iq_t + q_{z\bar{z}} - 2uq + 2(\bar{p}q)_z - 2pq_{\bar{z}} - 4|p|^2 q = 0 \\ ir_t - r_{z\bar{z}} + 2ur + 2(\bar{p}r)_z - 2pr_{\bar{z}} + 4|p|^2 r = 0 \\ ip_t + (qr)_{\bar{z}} - u_z = 0 \\ \bar{p}_z + p_{\bar{z}} = -|q|^2 + |r|^2, \ \ -\bar{r}_z + q_{\bar{z}} = -2(p\bar{r} + \bar{p}q), \end{cases}$$

where z is a complex coordinate of the plane $\mathbf{R}^2$ and $\bar{z}$ is the complex conjugate of z. Though this nonlinear Schrödinger-type system looks much complicated, it admits a class of the following explicit blowing-up smooth solutions:

$$p = 0, \ q == \frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}, \ r = \frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{(a+bt)}\alpha\bar{z}, \ u = \frac{2\alpha^2 z\bar{z}}{(a+bt)^2},$$

where a, b are real numbers with $ab < 0$ and α satisfies $\alpha^2 = \frac{b^2}{16}$. From these facts, we explicitly construct smooth solutions to the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ by using the gauge transformations, such that the absolute of their gradients blows up in finite time. This reveals some blow-up phenomenon of Schrödinger maps.

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§1. Introduction

The Landau-Lifshitz equations, or in other words, the generalized Heisenberg models for a continuous ferromagnetic spin vector $s = (s_1, s_2, s_3) \in S^2 \hookrightarrow \mathbf{R}^3$ (see, for example,

[12, 14, 17]),

$$\mathbf{s}_t = \mathbf{s} \times \Delta_{\mathbf{R}^n} \mathbf{s}, \quad x \in \mathbf{R}^n, \quad n = 1, 2, 3, \cdots, \tag{1}$$

where $\times$ denotes the cross product in the Euclidean 3-space $\mathbf{R}^3$ and $\Delta_{\mathbf{R}^n}$ is the Laplacian operator on $\mathbf{R}^n$, are important equations appeared in magnetic fields theory. Eqs.(1) have been generalized to various Hermitian symmetric spaces (see, for example [19]). The following is one of them

$$\mathbf{s}_t = \mathbf{s} \dot{\times} \Delta_{\mathbf{R}^n} \mathbf{s}, \quad x \in \mathbf{R}^n, \quad n = 1, 2, 3, \cdots, \tag{2}$$

where $\mathbf{s} = (s_1, s_2, s_3) \in \mathbf{H}^2 = \{(s_1, s_2, s_3) \in \mathbf{R}^{2+1} : |s|^2 = s_1^2 + s_2^2 - s_3^2 = -1, s_3 < 0\} \hookrightarrow \mathbf{R}^{2+1}$ is a 'unit' spin vector in the Minkowski 3-space $\mathbf{R}^{2+1}$, and $\dot{\times}$ denotes the pseudo-cross product in $\mathbf{R}^{2+1}$, i.e. for $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbf{R}^{2+1}$, $\mathbf{a} \dot{\times} \mathbf{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, -(a_1 b_2 - a_2 b_1))$. Eqs.(2) are regarded as dual versions of Eqs.(1) (see [11, 13]). In fact, Eqs.(1) relate to the $SU(2)$ ferromagnetic spin models and, meanwhile, Eqs.(2) relate to the $SU(1,1)$ ferromagnetic spin models in physics (see, for example [15]). The significance of studying Eq.(2) with $n = 2$ is also mentioned in [13]. We would like to point out that there are two components of the surface $s_1^2 + s_2^2 - s_3^2 = -1$ in $\mathbf{R}^{2+1}$, i.e., $s_3 > 0$ and $s_3 < 0$. Without loss of generality, we shall fix $\mathbf{H}^2$ as the above surface in $\mathbf{R}^{2+1}$ with the component of $s_3 > 0$, unless otherwise stated.

The Landau-Lifshitz equations (1) and their dual versions (2) are special cases and typical examples of the so-called Schrödinger maps ([2], [11], [13]) or Schrödinger flows ([8], [4]) in geometry. A Schrödinger map u from a Riemannian manifold (M, g) to a Kähler manifold (N, J) is defined to be a solution to the (infinite dimensional) Hamiltonian system of the energy function $E(u) = \int_M |\nabla u|^2 dv_g$ on the mapping space $C^k(M, N)$ for some $k > 0$. More explicitly, the equation of the Schrödinger maps $u : M \to N$ is expressed as the following evolution system:

$$u_t = J(u)\tau(u),$$

where J is the complex structure on N and $\tau(u)$ is the tension field of u. It is a straightforward verification that the Schrödinger map from the Euclidean n-space $\mathbf{R}^n$ to the 2-sphere $S^2 \hookrightarrow \mathbf{R}^3$ is just Eq.(1) (for example see [2], [8]) and the Schrödinger map from the Euclidean n-space $\mathbf{R}^n$ to the hyperbolic 2-space $\mathbf{H}^2 \hookrightarrow \mathbf{R}^{2+1}$ is nothing else than Eq.(2) ([4]).

Though Eqs.(1) (resp. Eqs.(2)) have a unified expressions for $n \geq 1$, there are great differences between dynamical properties of Eq.(1) (resp. Eq.(2)) with $n = 1$ and those of Eqs.(1) (resp. Eqs.(2)) with $n \geq 2$. When $n = 1$, Eq.(1) and (2) are integrable and can be solved by the method of inverse scattering techniques ([9]). Moreover, it was proved by Zarkharov and Takhtajan in [22] that Eq.(1) is gauge equivalent to the integrable focusing nonlinear Schrödinger equation: $i\phi_t + \phi_{xx} + 2|\phi|^2\phi = 0$ and, meanwhile, it was showed in [4] that Eq.(2) with $n = 1$ is gauge equivalent to the defocusing nonlinear Schrödinger equation: $q_t + q_{xx} - 2|q|^2 q = 0$. When $n \geq 2$, Eqs.(1) or Eqs.(2) are, roughly speaking, non-integrable and the understanding of their dynamical properties becomes

much more difficult than that of their one dimensional cases. The local in time existence and uniqueness of $W^{m,\sigma}(\mathbf{R}^n)$-solutions to the Cauchy problem of the Schrödinger map from $\mathbf{R}^n$ ($n \geq 2$) to 2-sphere $S^2 \hookrightarrow R^3$ was established by Sulem, Sulem and Bardos in 1986 in [17], where $\sigma \geq 2$. They also proved the global in time $W^{m+1,6}(\mathbf{R}^n)$-existence of the Cauchy problem of Eq.(1) for small initial data. In 2000, Chang, Shatah and Uhlenbeck displayed in [2] the global in time $W^{2,4}(\mathbf{R}^2)$-existence of radially symmetric solutions to the Cauchy problem of Schrödinger maps from $\mathbf{R}^2$ to Riemannian surfaces for small initial data. There is also previous work [8] by W.Y. Ding and Wang in local time existence and uniqueness of solutions for Schrödinger maps to Kähler manifolds. Grillakis and Stefanopoulos displayed in [11] conservation laws and localized energy estimates of Schrödinger maps to Riemannian surfaces. Recently, Nahmod, Stefanov and Uhlenbeck proved the local well-posedness of the Cauchy problem of the Schrödinger maps from $\mathbf{R}^2$ to the 2-sphere $\mathbf{S}^2$ and the hyperbolic 2-space $\mathbf{H}^2$ by using their equivalent equations which are so-called the modified Schrödinger map equations in [13]. However, it is widely believed that there should exist blowing-up solutions to Schrödinger maps (or flows) (see, for example, [7]) when the dimension of the starting manifolds is greater than 1.

In this paper, by using the geometric concept of gauge equivalence for PDEs with prescribed curvature representation applied in [5, 6], we show that the equation of the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2 \hookrightarrow \mathbf{R}^{2+1}$: $\mathbf{s}_t = \mathbf{s}\dot{\times}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\mathbf{s}$ is gauge equivalent to the following $1+2$ dimensional nonlinear Schrödinger-type system:

$$\begin{cases} iq_t + q_{z\bar{z}} - 2uq + 2(\bar{p}q)_z - 2pq_{\bar{z}} - 4|p|^2 q = 0 \\ ir_t - r_{z\bar{z}} + 2ur + 2(\bar{p}r)_z - 2pr_{\bar{z}} + 4|p|^2 r = 0 \\ ip_t + (qr)_{\bar{z}} - u_z = 0 \end{cases} \tag{3}$$

where $z = \frac{x_1 + ix_2}{2}$ is a formal complex version of the variables x_1 and x_2 ($\bar{z}$ is the complex conjugate of z), u is a unknown real function and p, q, r are unknown complex functions satisfying the following additional restrictions

$$\bar{p}_z + p_{\bar{z}} = -|q|^2 + |r|^2, \quad -\bar{r}_z + q_{\bar{z}} = -2(p\bar{r} + \bar{p}q). \tag{4}$$

This nonlinear Schrödinger-type system (3,4) is somewhat different from the modified Schrödinger map equations deduced by Nahmod, Stefanov and Uhlenbeck in [13]. By fixing an ansatz solution: $p = 0, q = e^{-i\theta}Q(\rho, t), r = e^{-i\theta}\bar{Q}(\rho, t), u = |Q|^2 - 2\int_\rho^\infty \frac{|Q|^2(\tau,t)}{\tau}d\tau$ for some functions $Q(t, \rho)$ to the system (3,4), where (ρ, θ) are the polar coordinates of (x_1, x_2), (3) leads $Q(t, \rho)$ to satisfy

$$iQ_t + \left(Q_{\rho\rho} + \frac{1}{\rho}Q_\rho - \frac{1}{\rho^2}Q\right) - 2Q\left(|Q|^2 - 2\int_\rho^\infty \frac{|Q(\tau, t)|^2}{\tau}d\tau\right) = 0. \tag{5}$$

This is a well-known nonlinear Schrödinger-type equation gauged to the radially symmetric Schödinger map from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ (see [2, 3]). As applications, by using the gauged nonlinear Schrödinger-type system (3,4), we first prove that there are no nontrivial smooth radially symmetric harmonic maps from $\mathbf{R}^2$ to the Hyperbolic

2-Space $\mathbf{H}^2$. Then we show the existence of class of smooth blowing-up solutions to the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ by constructing explicit blowing-up solutions to the nonlinear Schrödinger-type system (3,4). The latter reveals the blow-up phenomenon of Schrödinger maps and gives an affirmative answer to a problem proposed by W.Y. Ding in [7] for Schrödinger flows. This also indicates that the introduction of the gauged nonlinear Schrödinger-type system (3,4) is much more effective than that of the known Eq.(5) in the study of the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$.

This paper is organized as follows. In the section 2, we shall transform the equation of the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ to its equivalent nonlinear Schrödinger-type system (3,4) by applying geometric gauge theory. In section 3, as one of applications, we show the existence of class of smooth blowing-up solutions to the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ via the explicit construction of blowing-up solutions to the gauged nonlinear Schrödinger-type system (3,4). Finally, in section 4, we close the paper with some conclusions and remarks. We set an appendix at the end of the paper to give a detailed proof of explicit general solutions to a matrix equation which is important but not proved in the context of the paper.

§2. Gauge Equivalence

Zakharov and Takhtajan introduced in [22] the geometric concept of gauge equivalence between two soliton equations with zero curvature representation, which provides a useful tool in the study of integrable equations [9]. In [5, 6] the geometric concept of gauge equivalence between (integrable) differential equations with zero curvature representation has been generalized to (nonintegrable or integrable) differential equations with prescribed curvature representation. Now we find that this geometric idea is applicable to the present situation, as we shall see below.

It is obvious that the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2 \hookrightarrow \mathbf{R}^{2+1}$ (refer to (2) with $n = 2$) is equivalent to the following system:

$$\mathbf{s}_t = -\mathbf{s}\dot{\times}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\mathbf{s} \tag{6}$$

by $\mathbf{s} \to -\mathbf{s}$. Notice that, in this case, the surface $\mathbf{H} = \{s_1^2 + s_2^2 - s_3^2 = -1\}$ in $\mathbf{R}^{2+1}$ with $s_3 > 0$ becomes now the surface with $s_3 < 0$. Eq.(6) reads explicitly:

$$\begin{aligned}
(s_1)_t &= s_3(s_2)_{z\bar{z}} - s_2(s_3)_{z\bar{z}} \\
(s_2)_t &= s_1(s_3)_{z\bar{z}} - s_3(s_1)_{z\bar{z}} \\
(s_3)_t &= s_1(s_2)_{z\bar{z}} - s_2(s_1)_{z\bar{z}},
\end{aligned}$$

where $z = \frac{x_1+ix_2}{2}$, $\bar{z} = \frac{x_1-ix_2}{2}$ which is different from the usual complex version of the variables (x_1, x_2). Now we convert Eq.(6) to its matrix form:

$$S_t = -\frac{1}{2}[S, S_{z\bar{z}}], \tag{7}$$

where $S = \begin{pmatrix} is_3 & s_1 + is_2 \\ s_1 - is_2 & -is_3 \end{pmatrix} \in su(1,1)$ with $S^2 = -I$. Let's set

$$A = Vd\bar{z} + i\lambda S dz + i\lambda\Big(2V + S_{\bar{z}}S + 2\alpha S\Big)dt \tag{8}$$

where λ is a spectral parameter which is independent of t,z and $\bar{z}$, $V = V(\lambda, \xi, \eta, t)$ is a 2×2-matrix function satisfying the equation:

$$(i\lambda S)_{\bar{z}} - V_z + [i\lambda S, V] = 0 \tag{9}$$

and $\alpha = -\frac{i}{2}\mathrm{tr}(G^{-1}G_{\bar{z}}\sigma_3)$ is a function depending only on S, here G is an $SU(1,1)$-matrix defined by (13) below. $d + A$ can be geometrically interpreted as defining a connection on the trivial principal bundle $\mathbf{R}^3 \times SU(1,1)$ over $\mathbf{R}^3$ (the space of the independent variables x, y and t). From Yang-Mills theory, it is a straightforward computation that the curvature F_A of the connection (8) is

$$\begin{aligned} F_A &= dA - A \wedge A \\ &= \Big\{ - V_z + i\lambda S_{\bar{z}} - [V, i\lambda S]\Big\} d\bar{z} \wedge dz \\ &\quad + \Big\{ - V_t + i\lambda(2V + S_{\bar{z}}S + 2\alpha S)_{\bar{z}} - [V, i\lambda(2V + S_{\bar{z}}S + 2\alpha S)]\Big\} d\bar{z} \wedge dt \\ &\quad + \Big\{ - i\lambda S_t + i\lambda(2V + S_{\bar{z}}S + 2\alpha S)_z - [i\lambda S, i\lambda(2V + S_{\bar{z}}S + 2\alpha S)]\Big\} dz \wedge dt. \end{aligned}$$

Thus the choice of V in (9) above is to let the coefficient of $d\bar{z}\wedge dz$ be vanish. Furthermore, set

$$\begin{aligned} K &= \Big\{ - V_t + i\lambda(2V + S_{\bar{z}}S + 2\alpha S)_{\bar{z}} - [V, i\lambda(2V + S_{\bar{z}}S + 2\alpha S)]\Big\} d\bar{z} \wedge dt \\ &\quad + i\lambda\Big(\frac{1}{2}[S_{\bar{z}}, S_z] + 2(\alpha S)_z\Big) dz \wedge dt, \end{aligned} \tag{10}$$

then it is also a direct calculation that the following prescribed curvature condition:

$$F_A = dA - A \wedge A = K \tag{11}$$

is actually equivalent to the matrix equation (7). Here in the calculation we have used the fact $S^2 = -I$ and hence the identities: $S_{\bar{z}}S = -SS_{\bar{z}}$, $S_{\bar{z}z}S + S_{\bar{z}}S_z + S_z S_{\bar{z}} = -SS_{\bar{z}z}$ by taking derivatives.

In this section, our aim is to use the prescribed curvature representation formula (11) to transform the matrix equation (7) into the (1+2) dimensional nonlinear Schrödinger-type system (3,4) in the category of Yang-Mills theory.

Theorem 1 *For any given solution $S(t, z, \bar{z})$ to the Schrödinger map from $\mathbf{R}^2$ to $\mathbf{H}^2$ (7), there is a matrix-valued function $G(t, z, \bar{z}) \in SU(1,1)$ such that S is transformed to a solution $(p(t, z, \bar{z}), q(t, z, \bar{z}), r(t, z, \bar{z}), u(t, z, \bar{z}))$ to the 1+2 dimensional nonlinear Schrödinger-type system (3,4) by the gauge transformation with G.*

Proof. Let $S = S(t, z, \bar{z})$ be a solution to Eq.(7). We come to choose an $SU(1,1)$ matrix $G(t, z, \bar{z})$ such that

$$\sigma_3 = iG^{-1}SG, \quad G^{-1}G_z = -\begin{pmatrix} p & q \\ r & -p \end{pmatrix} := -U \tag{12}$$

for some complex functions $p = p(t, z, \bar{z})$ and $q = q(t, z, \bar{z})$ and $r = r(t, z, \bar{z})$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Pauli matrix. It is well-known that the general $SU(1,1)$-solutions of the matrix equation $\sigma_3 = iG^{-1}SG$ are of the from (see the proof in the appendix at the end of this paper):

$$G = \frac{1}{\sqrt{2(1-s_3)}}(S - i\sigma_3)\mathrm{diag}(\gamma, \bar{\gamma}), \tag{13}$$

where γ is a complex function of z, $\bar{z}$ and t with $|\gamma| = 1$. For such a $SU(1,1)$-matrix G given in (13), we have

$$G_{x_1} = -G\begin{pmatrix} is & \psi \\ \bar{\psi} & -is \end{pmatrix}, \quad G_{x_2} = -G\begin{pmatrix} il & \phi \\ \bar{\phi} & -il \end{pmatrix} \tag{14}$$

for some real functions s, l and complex functions ϕ, ψ. Since for the complex variables $z = (x_1 + ix_2)/2$ and $\bar{z} = (x_1 - ix_2)/2$, we have $\frac{\partial}{\partial z} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}$ and $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$. Hence, from (14),

$$G_z = -G\begin{pmatrix} l + is & (\psi - i\phi) \\ (\bar{\psi} - i\bar{\phi}) & -(l + is) \end{pmatrix}, \quad G_{\bar{z}} = -G\begin{pmatrix} -l + is & \psi + i\phi \\ (\bar{\psi} + i\bar{\phi}) & l - is \end{pmatrix}.$$

We thus have (12) with $p = l + is$, $q = \psi - i\phi$ and $r = \bar{\psi} - i\bar{\phi}$, and

$$G_{\bar{z}} = GP := G\begin{pmatrix} \bar{p} & -\bar{r} \\ -\bar{q} & -\bar{p} \end{pmatrix}. \tag{15}$$

The integrability condition $P_z + U_{\bar{z}} + [P, U] = 0$ of the linear system: $G_z = -GU, G_{\bar{z}} = GP$ implies

$$\bar{p}_z + p_{\bar{z}} = |r|^2 - |q|^2, \quad -\bar{r}_z + q_{\bar{z}} = -2(p\bar{r} + \bar{p}q),$$

which are exactly the restrictions (4). Furthermore, we have $\alpha = -i\bar{p}$ from (15) in the definition of the connection A given by (8). Now, for the connection A given in (8) with S being fixed above, we make the following gauge transformation:

$$\begin{aligned} A^G &= -G^{-1}dG + G^{-1}AG \\ &= -G^{-1}(G_{\bar{z}}d\bar{z} + G_z dz + G_t dt) + G^{-1}\left(V d\bar{z} + i\lambda S dz + i\lambda(2V + S_{\bar{z}}S + 2\alpha S)dt\right)G. \end{aligned} \tag{16}$$

We would like to determine this A^G explicitly and then prove the theorem in this process. In fact, from the relations (12,15) we have $S=-iG\sigma_3G^{-1}$, $S_{\bar z}=-2iGP^{(\text{off-diag})}\sigma_3G^{-1}$ and also $\alpha=-i\bar p$. Hence

$$\begin{aligned} S_{\bar z}S+2\alpha S &= -2iGP^{(\text{off-diag})}\sigma_3G^{-1}(-iG\sigma_3G^{-1})-2i\bar p(-iG\sigma_3G^{-1}) \\ &= -2G[P^{(\text{off-diag})}+\bar p\sigma_3]G^{-1}=-2G_{\bar z}G^{-1}. \end{aligned} \tag{17}$$

Thus substituting it into (16) and using $G_z=-GU$, we obtain

$$\begin{aligned} A^G &= (-G^{-1}G_{\bar z}+G^{-1}VG)d\bar z+(\lambda\sigma_3+U)dz \\ &\quad +\Big(-G^{-1}G_t+2i\lambda(G^{-1}VG-G^{-1}G_{\bar z})\Big)dt. \end{aligned} \tag{18}$$

As A satisfying the prescribed curvature condition: $F_A=dA-A\wedge A=K$, where K is given by (10), from the gauge theory we know that A^G must fulfill

$$F_{A^G}=dA^G-A^G\wedge A^G=G^{-1}(dA-A\wedge A)G=G^{-1}KG. \tag{19}$$

From (18) and by a direction computation we have

$$\begin{aligned} F_{A^G} &= \Big\{(G^{-1}G_{\bar z}-G^{-1}VG)_z+(\lambda\sigma_3+U)_{\bar z}-[-G^{-1}G_{\bar z}+G^{-1}VG,\lambda\sigma_3+U]\Big\}d\bar z\wedge dz \\ &\quad +\Big\{-(-G^{-1}G_{\bar z}+G^{-1}VG)_t+(-G^{-1}G_t+2i\lambda(G^{-1}VG-G^{-1}G_{\bar z}))_{\bar z} \\ &\quad -\Big[(-G^{-1}G_{\bar z}+G^{-1}VG),-G^{-1}G_t+2i\lambda(G^{-1}VG-G^{-1}G_{\bar z})\Big]\Big\}d\bar z\wedge dt \\ &\quad +\Big\{-(\lambda\sigma_3+U)_t+(-G^{-1}G_t+2i\lambda(G^{-1}VG-G^{-1}G_{\bar z}))_z \\ &\quad -\Big[\lambda\sigma_3+U,-G^{-1}G_t+2i\lambda(G^{-1}VG-G^{-1}G_{\bar z})\Big]\Big\}dz\wedge dt. \end{aligned}$$

Thus comparing respectively the coefficients of $dz\wedge d\bar z$, $d\bar z\wedge dt$ and $dz\wedge dt$ in the both sides of (19), we have the following identities:

$$-\Big(-G^{-1}G_{\bar z}+G^{-1}VG\Big)_z+U_{\bar z}-\Big[-G^{-1}G_{\bar z}+G^{-1}VG,\lambda\sigma_3+U\Big]=0, \tag{20}$$

$$\begin{aligned} &-\Big(-G^{-1}G_{\bar z}+G^{-1}VG\Big)_t+\Big(-G^{-1}G_t+i\lambda(2G^{-1}VG-2G^{-1}G_{\bar z})\Big)_{\bar z} \\ &-\Big[-G^{-1}G_{\bar z}+G^{-1}VG,-G^{-1}G_t+i\lambda(2G^{-1}VG-2G^{-1}G_{\bar z})\Big] \\ =\;& G^{-1}\Big\{-V_t+i\lambda(V_{\bar z}+S_{\bar z}S+2\alpha S)_{\bar z}-\Big[V,i\lambda(V_{\bar z}+S_{\bar z}S+2\alpha S)\Big]\Big\}G, \end{aligned} \tag{21}$$

and

$$
\begin{aligned}
& -U_t + \Big(- G^{-1}G_t + 2i\lambda(G^{-1}VG - G^{-1}G_{\bar z})\Big)_z \\
& -\Big[\lambda\sigma_3 + U, -G^{-1}G_t + 2i\lambda(G^{-1}VG - G^{-1}G_{\bar z})\Big] \\
= \; & G^{-1}i\lambda\Big(\frac{1}{2}[S_z, S_{\bar z}] + 2(\alpha S)_z\Big)G. \qquad (22)
\end{aligned}
$$

If we set

$$
\widetilde{V} = -G^{-1}G_{\bar z} + G^{-1}VG, \qquad (23)
$$

then (20) reads

$$
-\widetilde{V}_z + U_{\bar z} - [\widetilde{V}, \lambda\sigma_3 + U] = 0.
$$

On the one hand, from the relation $S = -iG\sigma_3G^{-1}$ we have

$$
\begin{aligned}
G^{-1}\Big((i\lambda S)_{\bar z} - V_z + [i\lambda S, V]\Big)G &= G^{-1}\Big(\lambda(iG\sigma_3G^{-1})_{\bar z} - V_z + [\lambda G\sigma_3G^{-1}, V]\Big)G \\
&= \lambda[G^{-1}G_{\bar z} - G^{-1}VG, \sigma_3] - G^{-1}V_zG. \qquad (24)
\end{aligned}
$$

On the other hand, from the relation $G^{-1}G_z = -U$ we also have

$$
\begin{aligned}
& U_{\bar z} - \widetilde{V}_z + [\lambda\sigma_3 + U, \widetilde{V}] \\
= \; & -(G^{-1}G_z)_{\bar z} - (-G^{-1}G_{\bar z} + G^{-1}VG)_z + [\lambda\sigma_3 - G^{-1}G_z, -G^{-1}G_{\bar z} + G^{-1}VG] \\
= \; & -G^{-1}V_zG + \lambda[G^{-1}G_{\bar z} - G^{-1}VG, \sigma_3]. \qquad (25)
\end{aligned}
$$

Hence, combine (24) with (25), we obtain

$$
G^{-1}\Big((i\lambda S)_{\bar z} - V_z + [i\lambda S, V]\Big)G = -\widetilde{V}_z + U_{\bar z} - [\widetilde{V}, \lambda\sigma_3 + U] = \text{left-hand-side of (20)},
$$

which implies that (20) is automatically satisfied from (9). We also claim that (21) is automatically satisfied. In fact, from the definition (23) of $\widetilde{V}$, we have $V = G\widetilde{V}G^{-1} + G_{\bar z}G^{-1}$ and hence, by the aid of (17),

$$
2V + S_{\bar z} + 2\alpha S = 2G\widetilde{V}G^{-1}.
$$

Thereofore

$$
\begin{aligned}
& \text{RHS of (21)} \\
= \; & G^{-1}\Big(- V_t + i\lambda(V_{\bar z} + S_{\bar z}S + 2\alpha S)_{\bar z} - \Big[V, i\lambda(V_{\bar z} + S_{\bar z}S + 2\alpha S)\Big]\Big)G \\
= \; & G^{-1}\Big(- (G\widetilde{V}G^{-1} + G_{\bar z}G^{-1})_t + 2i\lambda(G\widetilde{V}G^{-1})_{\bar z} - [G\widetilde{V}G^{-1} + G_{\bar z}G^{-1}, 2i\lambda G\widetilde{V}G^{-1}]\Big)G \\
= \; & -\widetilde{V}_t + (-G^{-1}G_t + 2i\lambda\widetilde{V})_{\bar z} - [\widetilde{V}, -G^{-1}G_t + 2i\lambda\widetilde{V}] \\
= \; & \text{LHS of (21)}.
\end{aligned}
$$

So nothing new is obtained from the two identities (20) and (21). Now we come to treat (22). By using the fact $\alpha = -i\bar{p}$, the first equation of (4) and the identities: $S = -iG\sigma_3 G^{-1}$, $S_z = 2iG(U^{(\text{off-diag})}\sigma_3)G^{-1}$ and $S_{\bar{z}} = 2iG\sigma_3 P^{(\text{off-diag})}G^{-1}$ deduced from (12) and (15) respectively, we have

$$\begin{aligned}
&\frac{1}{2}[S_{\bar{z}}, S_z] + 2(\alpha S)_z \\
&= G\left(2[U^{(\text{off-diag})}\sigma_3, \sigma_3 P^{(\text{off-diag})}] - 2\bar{p}_z\sigma_3 + 4\bar{p}U^{(\text{off-diag})}\sigma_3\right)G^{-1} \\
&= G\Big(2p_{\bar{z}}\sigma_3 + 4\bar{p}U^{(\text{off-diag})}\sigma_3\Big)G^{-1} = G\Big(2p_{\bar{z}}\sigma_3 + [\sigma_3, H]\Big)G^{-1},
\end{aligned} \tag{26}$$

where $H = \begin{pmatrix} 0 & -2q\bar{p} \\ -2r\bar{p} & 0 \end{pmatrix}$. Thus, (22) is equivalent to holding

$$-U_t + (-G^{-1}G_t + 2i\lambda\widetilde{V})_z - [\lambda\sigma_3 + U, -G^{-1}G_t + 2i\lambda\widetilde{V}] = 2i\lambda p_{\bar{z}}\sigma_3 + i\lambda[\sigma_3, H],$$

or equivalently,

$$-U_t + (-G^{-1}G_t)_z + [U, G^{-1}G_t] + \lambda\left(2iU_{\bar{z}} - 2ip_{\bar{z}}\sigma_3 + [\sigma_3, G^{-1}G_t] - i[\sigma_3, H]\right) = 0. \tag{27}$$

Here we have used the identities (20): $U_{\bar{z}} - \widetilde{V}_z + [\lambda\sigma_3 + U, \widetilde{V}] = 0$ in the computation. The vanishing of the coefficients of λ and the constant term in (27) lead to

$$-U_t + (-G^{-1}G_t)_z + [U, G^{-1}G_t] = 0, \tag{28}$$

$$2U_{\bar{z}} - 2p_{\bar{z}}\sigma_3 - i[\sigma_3, G^{-1}G_t] - [\sigma_3, H] = 0. \tag{29}$$

Since $G^{-1}G_t \in su(1,1)$, we may set $G^{-1}G_t = \begin{pmatrix} iu & \chi \\ \bar{\chi} & -iu \end{pmatrix}$ for some real function u and complex function χ. Then the equation (29) leads to $\chi = -i(q_{\bar{z}} + 2\bar{p}q)$ and $\bar{\chi} = i(r_{\bar{z}} - 2\bar{p}r)$ (the compatibility condition $\chi = \bar{\bar{\chi}}$ is equivalent to the second equation of (4)). Thus

$$G^{-1}G_t = i\begin{pmatrix} u & -q_{\bar{z}} - 2\bar{p}q \\ r_{\bar{z}} - 2\bar{p}r & -u \end{pmatrix} = i\left\{\left(u + U_{\bar{z}}^{(\text{off-diag})}\right)\sigma_3 + H\right\}. \tag{30}$$

Substituting (30) into the equation (28), we finally get the three nonlinear Schrödinger-type equations given by (3). This finishes the proof of the theorem. □

By the way, substituting (23) and (30) into (18) and (19) respectively, we obtain explicitly

$$A^G := \widetilde{V}d\bar{z} + \Big(\lambda\sigma_3 + U\Big)dz + \Big\{2i\lambda\widetilde{V} - i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 - iH\Big\}dt \tag{31}$$

and

$$
\begin{aligned}
K^G \quad := \quad & \Big\{ -\widetilde{V}_t + \Big(2i\lambda\widetilde{V} - i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 - iH\Big)_{\bar{z}} \\
& -[\widetilde{V}, 2i\lambda\widetilde{V} - i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 - iH]\Big\} d\bar{z} \wedge dt \\
& +i\lambda\Big(2U_{\bar{z}}^{(\text{diag})} + [\sigma_3, H]\Big) dz \wedge dt,
\end{aligned} \tag{32}
$$

where λ is the same spectral parameter as in (8), $U = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}$, $H = \begin{pmatrix} 0 & -2q\bar{p} \\ -2r\bar{p} & 0 \end{pmatrix}$ and $\widetilde{V}$ solves the equation

$$
U_{\bar{z}} - \widetilde{V}_z + [\lambda\sigma_3 + U, \widetilde{V}] = 0. \tag{33}
$$

For the nonlinear Schrödinger-type system (3,4), as indicated in the proof of Theorem 1, it is a PDE with prescribed curvature representation:

$$
F_{\widetilde{A}} = d\widetilde{A} - \widetilde{A} \wedge \widetilde{A} = \widetilde{K}, \tag{34}
$$

where $\widetilde{A} = A^G$ and $\widetilde{K} = K^G$ are given in (31) and (32) respectively.

Next we shall prove that the above gauge transformation from the matrix equation (7) of the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$ to the nonlinear Schrödinger-type system (3,4) is in fact reversible.

Theorem 2 *For any C^2-solution (p, q, r, u) to the nonlinear Schrödinger-type system (3,4), there is a matrix C^2-function $G \in SU(1,1)$ such that (p, q, r, u) is the gauge transformed to a C^3-solution S to the equation (7) of the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$ by G. Moreover, if we require that the gauge matrix G satisfies $G|_{t=z=\bar{z}=0} = I$. Then any C^m-solution ($m \geq 2$) to the nonlinear Schrödinger-type system (3,4) corresponds uniquely to a C^{m+1}-solution to the equation (7) of the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$ and vice versa.*

Proof: Let (p, q, r, u) be a solution to Eqs.(3,4). From the proof of Theorem 1, we see that Eqs.(3,4) are in fact the integrability condition of the following linear system:

$$
G_z = -GU, \quad G_t = Gi\left((u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 + H\right), \tag{35}
$$

or equivalently,

$$
\left\{
\begin{aligned}
G_{x_1} &= -G\begin{pmatrix} i\mathrm{Im}p & \psi \\ \bar{\psi} & -i\mathrm{Im}p \end{pmatrix} \\
G_{x_2} &= -G\begin{pmatrix} i\mathrm{Re}p & \phi \\ \bar{\phi} & -i\mathrm{Re}p \end{pmatrix} \\
G_t &= G\begin{pmatrix} iu & -iq_{\bar{z}} - 2i\bar{p}q \\ ir_{\bar{z}} - 2i\bar{p}r & -iu \end{pmatrix}
\end{aligned}
\right. \tag{36}
$$

where $\psi = (q+\bar{r})/2$ and $\phi = i(q-\bar{r})/2$. It should be point out that Eq.(4) implies that the (right) coefficient matrix in righthand side of the third equation of (36) is also an $su(1,1)$-matrix. This indicates that the general solutions G to (36) (i.e. (35)) belong to the group $SU(1,1)$. Since the coefficient matrices in (36) are of C^1, we let $G \in SU(1,1)$ be a fundamental C^2-smooth solution to (35) (or equivalently (36)). We shall use it to make the following gauge transformation for the connection $\widetilde{A} = A^G$ given in (31) with (p,q,r,u) being given above:

$$\begin{aligned} A &= (dG)G^{-1} + G\widetilde{A}G^{-1} \\ &= \Big(G_{\bar{z}}G^{-1} + G\widetilde{V}G^{-1}\Big)d\bar{z} + \Big(G_zG^{-1} + G(\lambda\sigma_3 + U)G^{-1}\Big)dz \\ &\quad + \Big(G_tG^{-1} + (2i\lambda\widetilde{V} - i(u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 - iH)G^{-1}\Big)dt. \end{aligned} \tag{37}$$

We try to show that the 1-form A defined by (37) is exactly the connection of Eq.(7) given by (8) when S and α are suitably determined. In fact, substituting the coefficient $i\lambda S$ of dz of (8) into (37) and comparing the coefficients of λ of dz in the both sides of (37), we obtain

$$G_z = -GU, \quad S = -iG\sigma_3G^{-1} \quad (\text{hence} \quad S^2 = -I). \tag{38}$$

The first equation of (38) is automatically satisfied because of the first equation of (35). The second equation of (38) is regarded as defining S. Now, we have to prove that the coefficients of $d\bar{z}$ and dt of A defined by (37) are respectively the same coefficients of $d\bar{z}$ and dt of the connection given in (8), that is,

$$V = G_{\bar{z}}G^{-1} + G\widetilde{V}G^{-1}, \tag{39}$$

$$i\lambda\Big(2V + S_{\bar{z}}S + 2\alpha S\Big) = G_tG^{-1} + G\left(2i\lambda\widetilde{V} - i\Big((u + U_{\bar{z}}^{(\text{off-diag})})\sigma_3 + H\Big)\right)G^{-1}. \tag{40}$$

Eq.(39) can be regarded as defining V if we can show that such a V solves Eq.(9), i.e., for the S being given in (38) we must have

$$(i\lambda S)_{\bar{z}} - V_z + [i\lambda S, V] = 0. \tag{41}$$

The proof of (41) is a direct computation. Indeed, by using the expression of V given in (39) and the fact that G fulfills (36) (this equivalent to having (12), (15) and the third equation of (36)), we have

$$\begin{aligned} (i\lambda S)_{\bar{z}} - V_z + [i\lambda S, V] &= G\Big(-P_z - [P,U] - \widetilde{V}_z + [\lambda\sigma_3 + U, \widetilde{V}]\Big)G^{-1} \\ &= G\left(U_{\bar{z}} - \widetilde{V}_z + [\lambda\sigma_3 + U, \widetilde{V}]\right)G^{-1}. \end{aligned}$$

Here have have used the fact: $U_{\bar{z}} + P_z + [P,U] = 0$. Since $\widetilde{V}$ satisfies (33), this establishes (41). For proving (40), since G satisfies the second equation of (35), it is easy to see that (40) is equivalent to

$$2V + S_{\bar{z}}S + 2\alpha S = 2G\widetilde{V}G^{-1}.$$

Since $G\widetilde{V}G^{-1} = V - G_{\bar z}G^{-1}$ by (39), this equation can also be written as

$$S_{\bar z}S + 2\alpha S = -2G_{\bar z}G^{-1}. \tag{42}$$

Now we take $\alpha = -i\bar p$ which fulfills the requirement of α in the definition of the connection (8). Substituting $S = -iG\sigma_3 G^{-1}$, $\alpha = -i\bar p$ and applying $G_{\bar z} = GP$, we have

$$\begin{aligned}
(S_{\bar z}S + 2\alpha S) &= (-iG\sigma_3G^{-1})_{\bar z}(-iG\sigma_3G^{-1}) + 2(-i\bar p)(-iG\sigma_3G^{-1}) \\
&= -G\bar z G^{-1} + G\sigma_3 P\sigma_3 G^{-1} - 2\bar p G\sigma_3 G^{-1} \\
&= -G\bar z G^{-1} - GPG^{-1} = -2G_{\bar z}G^{-1}.
\end{aligned}$$

This proves (42) and hence (40). Thus we have proved that the two connections given by (37) and (8) respectively are actually the same one when $S = -iG\sigma_3G^{-1}$ and $\alpha = -i\bar p$. What's the remainder for us to do is to prove that the curvature formula

$$F_A = K = G\widetilde{K}G^{-1} = GF_{\widetilde{A}}G^{-1} \tag{43}$$

under the gauge transformation is satisfied too, where K is given by (10) and $\widetilde{K} = K^G$ is given by (32). In fact, on the one hand, we see that

$$\begin{aligned}
G\widetilde{K}G^{-1} = \; & G\bigg(\bigg\{ -\widetilde{V}_t + \Big(2i\lambda\widetilde{V} - i(u + U_{\bar z}^{(\text{off-diag})})\sigma_3 - iH\Big)_{\bar z} \\
& -[\widetilde{V}, 2i\lambda\widetilde{V} - i(u + U_{\bar z}^{(\text{off-diag})})\sigma_3 - iH]\bigg\} d\bar z \wedge dt \\
& +\lambda(2U_{\bar z}^{(\text{diag})} + [\sigma_3, H])dz \wedge dt\bigg)G^{-1}.
\end{aligned} \tag{44}$$

On the other hand, by using (39, 42) and (35), it is a straightforward calculation that the coefficient of $d\bar z \wedge dt$ in K given by (10) is

$$\begin{aligned}
& -V_t + i\lambda(2V + S_{\bar z}S + 2\alpha S)_{\bar z} - [V, i\lambda(2V + S_{\bar z}S + 2\alpha S)] \\
= \; & -V_t + 2i\lambda(G\widetilde{V}G^{-1})_{\bar z} - [V, 2i\lambda G\widetilde{V}G^{-1}] \\
= \; & -G_{t\bar z}G^{-1} + G_{\bar z}GG_tG^{-1} - G_t\widetilde{V}G^{-1} - G\widetilde{V}_tG^{-1} + G\widetilde{V}G^{-1}G_tG^{-1} + 2i\lambda G_{\bar z}\widetilde{V}G^{-1} \\
& +2i\lambda G\widetilde{V}_{\bar z}G^{-1} - 2i\lambda G\widetilde{V}G^{-1}G_{\bar z}G^{-1} - 2i\lambda(G_{\bar z}\widetilde{V}G^{-1} - G\widetilde{V}G^{-1}G_{\bar z}G^{-1}) \\
= \; & G\bigg\{ -\widetilde{V}_t + \Big(2i\lambda\widetilde{V} - i(u + U_{\bar z}^{(\text{off-diag})})\sigma_3 - iH\Big)_{\bar z} \\
& -[\widetilde{V}, 2i\lambda\widetilde{V} - i(u + U_{\bar z}^{(\text{off-diag})})\sigma_3 - iH]\bigg\}G^{-1}.
\end{aligned} \tag{45}$$

(44) and (45) indicate that the two coefficients of $d\bar z \wedge dt$ in the both sides of (43) are the same one. Meanwhile, the coefficient of $dz \wedge dt$ of K is

$$i\lambda\Big(\frac{1}{2}[S_z, S_{\bar z}] + 2(\alpha S)_z\Big) = i\lambda G\Big(2U_{\bar z}^{(\text{diag})} + [\sigma_3, H]\Big)G^{-1}, \tag{46}$$

Here we have used the identity: $\frac{1}{2}[S_{\bar{z}}, S_z] + 2(\alpha S)_z = G(2p_{\bar{z}}\sigma_3 + [\sigma_3, H])G^{-1}$, where $H = \begin{pmatrix} 0 & -2q\bar{p} \\ -2r\bar{p} & 0 \end{pmatrix}$ as before, which is deduced from the same argument in getting (26). (44) and (46) imply that the two coefficients of $dz \wedge dt$ in the both sides of (43) are also the same one. Thus we have proved the desired identity (43), which implies the holding of the prescribed curvature representation (11) by the gauge theory. This indicates that S defined by the second equation of (38) from the solution (p, q, r, u) to (3,4) satisfies the matrix (7) and hence the equation of the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$.

Since G is a solution to the linear first-order differential system (36), it is well-known from linear theory of differential equations that such a G is unique if we propose the initial condition $G|_{t=z=\bar{z}=0} = I$ on G. Under this circumstance, we see that a solution (p, q, r, u) to Eqs.(3,4) corresponds uniquely to a solution S to Eq.(7) by the gauge transformation and vice versa. Furthermore, because of the relation $S_z = 2iGU^{(\text{off-diag})}\sigma_3 G^{-1}$ and $S_{\bar{z}} = -2iGP^{(\text{off-diag})}\sigma_3 G^{-1}$ deduced from (38), the remainder part of the theorem is obviously true. □

Remark 1 *Theorem 1,2 reveal a closed relation between C^m-solutions S to the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$ and C^{m-1}-solutions (p, q, r, u) to the nonlinear Schrödinger-type system (3,4). As we shall see in the next section, the nonlinear Schrödinger-type system (3,4) will play an essential role for us to show the existence of blowing-up C^∞-solutions to the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$.*

Let us introduce the polar coordinates (ρ, θ) of $\mathbf{R}^2$, that is, $x_1 = \rho\cos\theta, x_2 = \rho\sin\theta$. Thus we have

$$\frac{\partial}{\partial z} = e^{-i\theta}\left(\frac{\partial}{\partial \rho} - \frac{i}{\rho}\frac{\partial}{\partial \theta}\right), \quad \frac{\partial}{\partial \bar{z}} = e^{i\theta}\left(\frac{\partial}{\partial \rho} + \frac{i}{\rho}\frac{\partial}{\partial \theta}\right).$$

We would like to find following ansatz solutions to (3,4):

$$p = 0, q = e^{-i\theta}Q(t,\rho), r = e^{-i\theta}\bar{Q}(t,\rho), u = |Q|^2(t,\rho) - 2\int_\rho^\infty \frac{|Q|^2(t,\tau)}{\tau}d\tau \tag{47}$$

for some suitable functions $Q(t, \rho)$. One may verify that (4) and the third equation of (3) are satisfied automatically (since $\partial_z^{-1}\partial_{\bar{z}}(rq) = \partial_z^{-1}\partial_{\bar{z}}(e^{-2i\theta}|Q|^2(t,\rho)) = |Q|^2(t,\rho) - 2\int_\rho^\infty \frac{|Q|^2(t,\tau)}{\tau}d\tau$) and the first and second equation of (3) lead Q to satisfy the nonlinear Schrödinger differential-integral equation (5) presented in Introduction. This equation was deduced in [2] by the (generalized) Hasimoto transformation. Furthermore, under this circumstance the linear system (36) is reduced to

$$\begin{cases} G_\rho = -G\begin{pmatrix} 0 & Q(t,\rho) \\ \bar{Q}(t,\rho) & 0 \end{pmatrix}, \\ G_t = Gi\begin{pmatrix} |Q|^2(t,\rho) - 2\int_\rho^\infty \frac{|Q|^2(t,\tau)}{\tau}d\tau & -Q_\rho(t,\rho) - Q(t,\rho)/\rho \\ \bar{Q}_\rho(t,\rho) + \bar{Q}(t,\rho)/\rho & -|Q|^2(t,\rho) + 2\int_\rho^\infty \frac{|Q|^2(t,\tau)}{\tau}d\tau \end{pmatrix}. \end{cases} \tag{48}$$

So the gauge matrix function $G(z,\bar{z},t)=G(t,\rho)$ is radial. The corresponding matrix S given by the second equation (38) is exactly $-iG(t,\rho)\sigma_3G^{-1}(t,\rho)$ and hence is a radially symmetric solution to Eq.(7) of the Schrödinger flow from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$.

§3. Applications

We will follow the basic conventional notations for Sobolev spaces $W^{m,\sigma}(\mathbf{R}^n)$ ($\sigma\geq 2$) of real or complex-valued functions or spaces $C^k(\mathbf{R}^n)$ of continuous differential functions up to order k on $\mathbf{R}^n$ for $n\geq 2$ and norms $||\cdot||_{W^{m,\sigma}(\mathbf{R}^n)}$ or $||\cdot||_{C^k(\mathbf{R}^n)}$ used in [10]

The gauged nonlinear Schrödinger-type system (3,4) looks much complicated and it seems seldom useful. However, the nonlinear Schrödinger-type system (3,4) is in fact another mathematical description of the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ and should be useful. In this section, as applications, we shall use the gauged nonlinear Schrödinger-type system (3,4) to explore some geometric property of harmonic maps from $\mathbf{R}^2$ to $\mathbf{H}^2$ and especially the blow-up property of the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$.

The first application is to study whether there exists a nontrivial smooth radially symmetric harmonic map from the whole plane $\mathbf{R}^2$ to $\mathbf{H}^2$. If there does exist a such harmonic map, then we may have a nontrivial radial solitonary wave solution (p,q,r,u) to (3,4) from it. Unfortunately, there is no such a nontrivial radial symmetric smooth harmonic maps from $\mathbf{R}^2$ to $\mathbf{H}^2$ defined on the whole plane $\mathbf{R}^2$.

Theorem 3 *There does not exist a nontrivial radially symmetric smooth harmonic map from the whole plane $\mathbf{R}^2$ to $\mathbf{H}^2$.*

Proof. If there exists a nontrivial radially symmetric smooth harmonic map from $\mathbf{R}^2$ to $\mathbf{H}^2$, then we denote it by $S=S(\rho)$, where (ρ,θ) is the polar coordinate of the plane. From the appendix at the end of the paper, we see that

$$G=\frac{1}{\sqrt{2(1-s_3)}}(S-i\sigma_3)\mathrm{diag}(\gamma,\bar{\gamma}),$$

is a smooth $SU(1,1)$-matrix function satisfying $\sigma_3=iG^{-1}SG$, where γ is a complex function of ρ and t with $|\gamma|=1$. Since $\frac{1}{\sqrt{2(1-s_3)}}(S-i\sigma_3)$ depends only on the radial variable ρ and is nontrivial, we may choose γ such that

$$G^{-1}G_\rho=-\begin{pmatrix}0 & q(t,\rho)\\ \bar{q}(t,\rho) & 0\end{pmatrix},\quad G^{-1}G_t=\begin{pmatrix}\gamma^{-1}\gamma_t & 0\\ 0 & \bar{\gamma}^{-1}\bar{\gamma}_t\end{pmatrix},\tag{49}$$

for some nonzero function $q(t,\rho)$ depending on ρ and possibly on t. On the other hand, from the proof of Theorem 1 we know such G should satisfy (48) for some smooth (nontrivial) $Q(t,\rho)$ too. Notice that the two matrices G satisfying respectively (49) and (48)

are at most up to a (smooth) diagonal matrix $\mathrm{diag}(\gamma_0, \bar{\gamma}_0)$ with $|\gamma_0| = 1$. Thus, compare (48) with (49), we have

$$Q(t,\rho) = \gamma_0^{-1} q(t,\rho)\bar{\gamma}_0, \quad Q_\rho + Q/\rho = 0. \tag{50}$$

Hence $Q(t,\rho) = \frac{C(t)}{\rho}$ for some nonzero constant $C(t)$ depending only on t. Since $Q(t,\rho)$ has singular at the origin except $C(t) = 0$, which is a contradiction to that $Q(t,\rho)$ and hence $q(t,\rho)$ is a nontrivial smooth function of t and ρ. This contradiction implies that $S = S(\rho)$ is not nontrivial smooth on the whole plane, which proves the theorem. □

The next application is to use the gauged nonlinear Schrödinger-type system (3,4) to construct a class of blowing-up solutions of the Schrödinger maps to the hyperbolic 2-space $\mathbf{H}^2$ (7). This is an interesting problem in the study of Schrödinger maps which was proposed by Y.W. Ding in [7] as a unsolved problem. In order to study such a problem, let's first give a brief description of some blowing-up results for nonlinear Schrödinger equations. It is well-known that, a nonlinear Schrödinger equation with the critical Sobolev exponential $\sigma = 1 + 4/n$ in $\mathbf{R}^n$: $iq_t + \Delta q + |q|^{\sigma-1} q = 0$ admits a so-called conformal invariance (see, for example, [21, 20, 18]). That says, the transformation from a solution q to

$$\tilde{q}(t,x) = \frac{e^{i\frac{b|x|^2}{4(a+bt)}}}{(a+bt)^{n/2}} q(T,X),$$

is invariant, where $X = \frac{x}{a+bt}$ with $x = (x_1, \cdots, x_n) \in \mathbf{R}^n$, $T = \frac{c+dt}{a+bt}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{R})$ (i.e. a, b, c, d are real and $ad - bc = 1$), i.e., $\tilde{q}(t,x)$ is still a solution to the same equation. Weinstein constructed in [21] blowing up solutions to the nonlinear Schrödinger equation with the critical exponential from solitonary wave solutions (see [1, 16] for the existence of such solitonary wave solutions) by using this conformal invariant property. Now for our present nonlinear Schrödinger-type system (3,4), the following lemma is very crucial. This fact is somewhat difficult to take note of, but it arises heuristically from the conformal invariant property of nonlinear Schrödinger equations mentioned above.

Lemma 1 *Let (p, q, r, u) be given as follows:*

$$\begin{aligned} p = p(t,z,\bar{z}) &= 0 \\ q = q(t,z,\bar{z}) &= \frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt} \alpha \bar{z} \\ r = r(t,z,\bar{z}) &= \frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{(a+bt)} \alpha \bar{z} \\ u = u(t,z,\bar{z}) &= \frac{2\alpha^2 z\bar{z}}{(a+bt)^2}, \end{aligned}$$

where a, b and α are real numbers. Then (p, q, r, u) is a solution to the gauged nonlinear Schrödinger system (3,4) if and only if $\alpha^2 = \frac{b^2}{16}$.

Proof. The proof is just a direct verification one by one:

$$-\bar{r}_z + q_{\bar{z}} + 2(p\bar{r} + \bar{p}q) = -\left(\frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha z\right)_z + \left(\frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha \bar{z}\right)_{\bar{z}} = 0,$$

$$\bar{p}_z + p_{\bar{z}} + |q|^2 - |r|^2 = \left|\frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}\alpha\bar{z}}{a+bt}\right|^2 - \left|\frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}\alpha\bar{z}}{a+bt}\right|^2 = 0,$$

and

$$\begin{aligned}
& iq_t + q_{z\bar{z}} - 2uq + 2(\overline{p}q)_z - 2pq_{\bar{z}} - 4|p|^2 q \\
&= i\left(\frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}\right)_t + \left(\frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}\right)_{z\bar{z}} - \frac{4\alpha^2 z\bar{z}}{(a+bt)^2}\left(\frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}\right) \\
&= \left(\frac{b^2}{2} - \frac{b^2}{4} - 4\alpha^2\right)\frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{(a+bt)^3}\alpha\bar{z}; \\
& ir_t - r_{z\bar{z}} + 2ur + 2(\overline{p}r)_z - 2pr_{\bar{z}} + 4|p|^2 r \\
&= i\left(\frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}\right)_t - \left(\frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}\right)_{z\bar{z}} + \frac{4\alpha^2 z\bar{z}}{(a+bt)^2}\left(\frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}\right) \\
&= \left(-\frac{b^2}{2} + \frac{b^2}{4} + 4\alpha^2\right)\frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{(a+bt)^3}\alpha\bar{z}; \\
& ip_t + (qr)_{\bar{z}} - u_z = \left(\frac{\alpha^2(\bar{z})^2}{(a+bt)^2}\right)_{\bar{z}} - \left(\frac{2\alpha^2 z\bar{z}}{(a+bt)^2}\right)_z = 0.
\end{aligned}$$

Since $\alpha^2 = \frac{b^2}{16}$, the lemma is thus proved. $\square$

It is obvious that solutions (p, q, r, u) constructed in Lemma 1 blow up in finite time (as $t \to -\frac{a}{b}$) at every point in the plane except the origin. We now prove the existence of blowing-up solutions to Eq.(7) from these solutions.

Theorem 4 *For real numbers a and b with $a > 0$ and $b < 0$, there exists a C^∞-solution $S(t, z, \bar{z})$ to the equation (7) of the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ such that the absolute of the gradient $|\nabla S|$ blows up in finite time at every point in $\mathbf{R}^2$ except the origin.*

Proof. For real numbers a and b with $a > 0$ and $b < 0$, we may construct the explicit solution (p, q, r, u) to the gauged nonlinear Schrödinger system (3,4) by

$$p = 0,\ q = \frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z},\ r = \frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{(a+bt)}\alpha\bar{z},\ u = \frac{2\alpha^2 z\bar{z}}{(a+bt)^2}$$

from Lemma 1, where $\alpha^2 = \frac{b^2}{16}$. By Theorem 2 there is a smooth solution $S(z, \bar{z}, t)$ to the Eq.(7) which is the $SU(1,1)$-gauge equivalent to the above solution (p, q, r, u) of (3,4) by a gauge matrix G. On the one hand, from the formulas: $G_z = -GU$, $G_{\bar{z}} = GP$ displayed in Theorem 1 or 2, where $U = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}$, $P = \begin{pmatrix} \overline{p} & \overline{r} \\ \overline{q} & -\overline{p} \end{pmatrix}$ and $G \in SU(1,1)$, we have

$$S_z = 2iGU^{(\text{off-diag})}\sigma_3 G^{-1}, \quad S_{\bar{z}} = -2iGP^{(\text{off-diag})}\sigma_3 G^{-1}.$$

On the other hand, from the appendix at the end of this paper we have that such G can be expressed as

$$G = \frac{1}{\sqrt{2(1-s_3)}}(S - i\sigma_3)\text{diag}(\gamma, \bar{\gamma}),$$

where γ is a complex function of z, $\bar{z}$ and t with $|\gamma| = 1$. Hence, the above two matrix identities read

$$(S - i\sigma_3)\text{diag}(\gamma, \bar{\gamma}) \begin{pmatrix} 0 & -q \\ r & 0 \end{pmatrix} = \frac{1}{2i} S_z (S - i\sigma_3)\text{diag}(\gamma, \bar{\gamma})$$

and

$$(S - i\sigma_3)\text{diag}(\gamma, \bar{\gamma}) \begin{pmatrix} 0 & \bar{r} \\ -\bar{q} & 0 \end{pmatrix} = \frac{1}{2i} S_{\bar{z}} (S - i\sigma_3)\text{diag}(\gamma, \bar{\gamma}).$$

Equaling the corresponding (row 1 and column 2) entries in the both sides of the matrix identities, we obtain

$$\begin{aligned} -i(s_3 - 1)q\gamma &= \frac{1}{2}\Big[(s_3)_z(s_1 + is_2) - ((s_1)_z + i(s_2)_z)(s_3 - 1)\Big]\bar{\gamma} \\ i(s_3 - 1)\bar{r}\gamma &= \frac{1}{2}\Big[(s_3)_{\bar{z}}(s_1 + is_2) - ((s_1)_{\bar{z}} + i(s_2)_{\bar{z}})(s_3 - 1)\Big]\bar{\gamma}. \end{aligned}$$

Hence

$$|q| + |r| \le |(s_1)_z| + |(s_1)_{\bar{z}}| + |(s_2)_z| + |(s_2)_{\bar{z}}| + |(s_3)_z| + |(s_2)_{\bar{z}}| := |\nabla S|.$$

Here we have used the facts: $|\gamma| = 1$, $|s_1/(1 - s_3)| \le 1$ and $|s_2/(1 - s_3)| \le 1$ because of $s_3^2 = 1 + s_1^2 + s_2^2$ and $s_3 < 0$. Since $q = \frac{e^{i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}$ and $r = \frac{e^{-i\frac{bz\bar{z}}{2(a+bt)}}}{a+bt}\alpha\bar{z}$ given above, we see that $|q| + |r| = 2\frac{|\alpha|\cdot|z|}{(a+bt)} \to +\infty$ as $t \to -\frac{a}{b}$ except $z = 0$. This implies that $|\nabla S|$ blows up at every point in the plane as $t \to -\frac{a}{b}$ except $z = 0$. The proof of the theorem is completed. □

Remark 2 *The collection of solutions $S(t, z, \bar{z})$ constructed in Theorem 4 composes a class of blowing-up C^∞-solutions to the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$. Since these solutions are constructed from explicit solutions (p, q, r, u) to the gauged nonlinear Schrödinger-type system (3,4) in Lemma 1, they are also called explicit blowing-up solutions to the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ in this paper.*

§4. Conclusion

In this paper, we have proved the existence of a class of smooth blowing-up solutions to the Schrödinger maps from $\mathbf{R}^2$ to the hyperbolic 2-space $\mathbf{H}^2$ with aid of its gauged equivalent nonlinear Schrödinger-type system (3,4). This sets a concrete example of the existence of blowing-up smooth solutions to Schrödinger maps, which answers a problem proposed in [7] for Schrödinger flows. It also indicates that the introduction of the gauged nonlinear Schrödinger-type system (3,4) is much more effective than that of the known Eq.(5) and we believe that this system will be much useful in exploring some deeper dynamical properties of the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$. By the way, in a completely similar way, one may transform the $1+2$ dimensional Landau-Lifshitz equation: $\mathbf{s}_t = \mathbf{s}\times\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}\right)\mathbf{s}$, where $s=(s_1,s_2,s_3)\in S^2\hookrightarrow\mathbf{R}^3$, into the following 1+2 dimensional nonlinear Schrödinger-type system

$$\left\{\begin{array}{l} iq_t - q_{z\bar z} + 2uq - 2(\bar p q)_z + 2pq_{\bar z} + 4|p|^2 q = 0 \\ ir_t + r_{z\bar z} - 2ur - 2(\bar p r)_z + 2pr_{\bar z} - 4|p|^2 r = 0 \\ ip_t = (qr)_{\bar z} - u_z \\ \bar p_z + p_{\bar z} = |q|^2 - |r|^2 \\ \bar r_z + q_{\bar z} = 2(p\bar r - \bar p q), \end{array}\right.$$

by an $SU(2)$-gauge matrix. From this system we may also prove that there are no non-trivial radially symmetric smooth harmonic maps from the whole $\mathbf{R}^2$ to the 2-sphere $\mathbf{S}^2$. But we are unable to establish a similar Lemma 1 for the present system and hence not to explicitly construct blowing-up smooth solutions to the Landau-Lifshitz equation.

Finally, we remark that the blowing-up smooth solutions to the Schrödinger maps from $\mathbf{R}^2$ to $\mathbf{H}^2$ constructed in this paper do not belong to any Sobolev space $W^{m,\sigma}(\mathbf{R}^2)$ (for a fixed t). It is very interesting that whether there exist blowing-up $W^{m,\sigma}(\mathbf{R}^2)$-solutions to Schrödinger maps, especially to the Landau-Lifshitz equations. However, this problem is still unknown.

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Appendix

This appendix gives a proof of the following fact used in the proofs of Theorem 1 and 4.

Proposition 1 *For a given matrix* $S=S(t,z,\bar z)=\begin{pmatrix} is_3 & s_1+is_2 \\ s_1-is_2 & -is_3\end{pmatrix}\in su(1,1)$, *where* $(s_1)^2+(s_2)^2-(s_3)^2=-1$ *with* $s_3<0$, *the general* $SU(1,1)$*-solutions* $G=G(t,z,\bar z)$

to the matrix equation $\sigma_3 = iG^{-1}SG$ are of the from:

$$G = \frac{1}{\sqrt{2(1-s_3)}}(S - i\sigma_3)\mathrm{diag}(\gamma, \bar{\gamma}),$$

where γ is a complex function of z, $\bar{z}$ and t with $|\gamma| = 1$.

Proof. Since a matrix in $SU(1,1)$ is of the form $\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$, where A and B are some complex numbers with $|A|^2 - |B|^2 = 1$, we may assume that $G = \begin{pmatrix} \nu & \chi \\ \bar{\chi} & \bar{\nu} \end{pmatrix}$, where ν and χ are unknown complex functions of z, $\bar{z}$ and t with $|\nu|^2 - |\chi|^2 = 1$. Substituting this matrix expression into the matrix equation $\sigma_3 = iG^{-1}SG$, we see that the matrix equation is equivalent to

$$\begin{cases} -(s_3+1)\nu + i(s_1 + is_2)\bar{\chi} = 0 \\ i(s_1 - is_2)\nu + (s_3 - 1)\bar{\chi} = 0. \end{cases} \tag{51}$$

Since $(s_1)^2 + (s_2)^2 - (s_3)^2 = -1$ and hence the rank of the coefficient matrix of the linear system (51) is 1, we have from linear algebra that the linear system (51) has nontrivial solutions and all the solutions compose an 1-dimensional linear (complex) space. Now it is a direct verification that a pair of $\nu = \nu_0 = i(s_3 - 1)/\sqrt{2(1-s_3)}$ and $\bar{\chi} = \bar{\chi}_0 = (s_1 - is_2)/\sqrt{2(1-s_3)}$ solves the linear system (51) and satisfies $|\nu_0|^2 - |\chi_0|^2 = 1$. So (ν_0, χ_0) can be regarded as a base of the linear (complex) space of solutions to (51). Hence the general solutions to (51) are

$$(\nu, \bar{\chi}) = \gamma(\nu_0, \bar{\chi}_0)$$

for a complex function $\gamma = \gamma(t, z, \bar{z})$ and the restriction of $|\nu|^2 - |\chi|^2 = 1$ implies $|\gamma| = 1$. Return to matrix G, we thus obtain that the general $SU(1,1)$-solutions to $\sigma_3 = iG^{-1}SG$ are

$$G = \frac{1}{\sqrt{2(1-s_3)}}(S - i\sigma_3)\mathrm{diag}(\gamma, \bar{\gamma}),$$

where γ is a complex function with $|\gamma| = 1$. This completes the proof of the proposition. $\square$

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